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Solutions of a Bethe–Salpeter Equation

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Abstract

We study systematically and exhibit the Lorentz symmetry of the Bethe–Salpeter equation in the light-like case for two scalar quarks of different masses interacting via the exchange of a scalar photon. We develop a new approach for solving the eigenvalue problem of the equation for the general case (not only the light-like). Our method permits accurate analytic expressions for the spectrum and the wave functions.

1. *Introduction*

The Bethe–Salpeter (BS) equation (Salpeter & Bethe, 1951) has been used extensively in the last years as a theoretical physics laboratory. In particular it has been used as a field theoretical model for studying the scattering amplitude and its symmetries. This led to the discovery of important new dynamical mechanisms and new concepts like that of Regge poles in relativistic field theory, Lorentz poles and daughter trajectories.†

The symmetries of the BS equation are well known for the vacuum-like, time-like and the space-like case. In the present paper we investigate systematically the symmetry for the light-like case, before and after the Wick rotation, i.e. in the original Minkowski and in Euclidean space. The eigenvalue problem is then reduced into a partial differential equation which exhibits the symmetry explicitly again. With polar coordinates our equation is further transformed into a one-dimensional ordinary differential equation.

† A general survey of the theory of the Bethe–Salpeter equation is given by Nakanishi (1969).

This equation can be transformed to another differential equation known in the mathematical literature as Heun's equation (Heun, 1889). The same approach when applied in the other cases, time-like, vacuum-like and space-like, leads to corresponding known one-dimensional differential equations which are also of Heun's type.

Analytic expressions of the eigenvalues of the general Heun's equation do not exist. Also little is known for the eigenfunctions of this equation (Bateman Manuscript Project, 1954). For this reason, up to now the eigenvalues of the BS equation and the Regge trajectories have been calculated mainly computationally (Chung & Snider, 1967). Such calculations, even when they are of great precision, hide many interesting features of the problem.

In our method the one-dimensional equation is transformed into an equivalent 'Schrödinger' equation. Then the potential is replaced by an approximate one, for which the resulting equation can be solved exactly. In this way we obtain analytic expressions in general very precise, for the eigenvalues and the eigenfunctions. For special values of the relative mass difference and/or the binding energy the original potential coincides with the approximate one, and we get exact solutions including all the known ones.

We hope that our approach will offer a more transparent picture of many qualitative and quantitative features of the spectrum, Regge trajectories and wave functions of the BS equation.

2. *Symmetries and Differential Equations*

We consider the Bethe–Salpeter equation of two scalar quarks which form a bound state via the exchange of a scalar particle (Wick–Cutkosky model (Wick, 1954; Cutkosky, 1954)). Let m_i, $p_{i\mu}$, $i = 1, 2$ be the masses and the 4-momenta of the quarks 1 and 2. We write

$$\begin{aligned} m_1 &= m(1+\Delta) \\ m_2 &= m(1-\Delta) \end{aligned} \tag{2.1}$$

and we introduce the total momentum p of the bound state ($p^2 = s$), and the relative momentum q of the quarks by

$$\begin{aligned} p_1 &= q + \tfrac{1}{2}(1+\Delta)p \\ p_2 &= -q + \tfrac{1}{2}(1-\Delta)p \end{aligned} \tag{2.2}$$

Then the BS equation takes the form (Wick, 1954)

$$\{[q+\tfrac{1}{2}(1+\Delta)p]^2 - (1+\Delta)^2\}\{[q-\tfrac{1}{2}(1-\Delta)p]^2 - (1-\Delta)^2\}\,\Phi(q,p) = -\frac{i\lambda}{\pi^2}\int \frac{d^4k}{(q-k)^2-\mu^2}\,\Phi(k,p) \tag{2.3}$$

where μ is the mass of the exchanged particle, and the mass units were chosen such that $m = 1$. We make a stereographic projection of the four-dimensional space into the surface of a sphere in five-dimensional space, i.e. we introduce new variables (Cutkosky, 1954; Delbourgo *et al.*, 1966)

$$\eta_\mu = \frac{2q_5 q_\mu}{q_5^2 - q^2}, \qquad \mu = 0, 1, 2, 3$$

$$\eta_5 = \frac{q_5^2 + q^2}{q_5^2 - q^2}$$

$$\eta_i \eta^i = \eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_2^2 - \eta_5^2 = -1 \tag{2.4}$$

where

$$q_5 = \left[\left|1 - \frac{s}{4}\right|(1 - \Delta^2)\right]^{1/2} \tag{2.5}$$

In the η variables equation (2.3) is put in a compact form

$$[q_5^2 - \tfrac{1}{4}(p_i' \eta^i)^2]\Psi(\eta_i, p) = \frac{i\lambda(1 - \Delta^2)}{4\pi^2} \int \frac{d^5\eta' \, \delta(\eta_i' \eta'^i + 1)\Psi(\eta_2', p)}{1 + \eta_i' \eta^i + \dfrac{\mu^2}{2q_5^2}(1 + \eta_5)(1 + \eta_5')}$$

$$i = 0, \ldots 3, 5 \tag{2.6}$$

where

$$\Psi(\eta_i, p) = (1 + \eta_5)^{-3} \Phi(q, p) \tag{2.7}$$

$$p_i' = [p_\mu(1 - \Delta^2), 2q_5 \Delta] \tag{2.8}$$

The metric in the five-dimensional scalar products of equation (2.6) is

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{55} = 1$$

Equation (2.3) for $\mu = 0$ can be easily transformed to a differential equation, after the Wick rotation. With the use of the identity

$$\frac{\partial^2}{\partial q_\alpha \partial q_\alpha} \frac{1}{(q - k)^2} = -4\pi^2 \delta^{(4)}(q - k) \tag{2.9}$$

the integral equation (2.3) becomes

$$(\Box\{[q + \tfrac{1}{2}(1 + \Delta)p]^2 + (1 + \Delta)^2\} \times \{[q - \tfrac{1}{2}(1 - \Delta)p]^2 + (1 - \Delta)^2\} - 4\lambda)\Phi(q, p) = 0 \tag{2.10}$$

With a change of variables (Kyriakopoulos, 1968)

$$z_i = \eta_i z, \qquad z = p_5, \qquad i = 1, \ldots, 5 \tag{2.11}$$

and

$$Y(\eta_i, p) = \left[1 - \frac{1}{4z^4}(p_i' z_i)^2\right]\left(1 + \frac{z_5}{z}\right)^{-3} \Phi(q, p) \tag{2.12}$$

equation (2.10) becomes

$$\left\{\left[1-\frac{1}{4z^4}(p_i' z_i)^2\right]\left[z^2\frac{\partial^2}{\partial z_i^2}-\left(z_i\frac{\partial}{\partial z_i}\right)^2-3z_i\frac{\partial}{\partial z_i}-2\right]-\frac{\lambda}{1-(s/4)}\right\} \times Y(z_i,p)=0 \qquad (2.13)$$

The above equation is the BS equation in differential form.

2.1. *Light-Like Case*

In a coordinate system in which the total energy-momentum vector is $p_\mu=(\omega,0,0,\omega)$, equation (2.6) reads

$$\{1-[a_1(\eta_0-\eta_3)-\Delta\eta_5]^2\}\Psi(\eta_i,p) = \frac{i\lambda}{4\pi^2}\int\frac{d^5\eta'\,\delta(\eta_i'\eta'^i+1)\Psi(\eta_i',p)}{1+\eta_i'\eta^i+\dfrac{\mu^2}{2q_5^2}(1+\eta_5)(1+\eta_5')} \qquad (2.1.1)$$

where

$$a_1=\frac{\omega\surd(1-\Delta^2)}{2} \qquad (2.1.2)$$

For $\mu\neq 0$ the symmetry group of equation (2.1.1) is obviously the E_2. We want to investigate the group of invariance of the equation for $\mu=0$. In this case the symmetry group will be the subgroup of SO(4,1) which leaves invariant the expression

$$\eta_5-\frac{\alpha_1}{\Delta}(\eta_0-\eta_3) \qquad (2.1.3)$$

Let L_{ij}, $i,j=0,\ldots,3,5$ be the generators of the group SO(4,1). Considering infinitesimal transformations we find that the expression (2.1.3) remains invariant if the generators appear only in the combinations

$$L_{13}-L_{01},\qquad L_{23}-L_{02},\qquad L_{15}-\frac{\Delta}{a_1}L_{01},\qquad L_{25}-\frac{\Delta}{a_1}L_{02},$$

$$L_{35}+L_{05}-\frac{\Delta}{a_1}L_{03},\qquad \text{and}\qquad L_{12}$$

By taking linear combinations of them we get the operators

$$J_{12}=L_{12}$$

$$J_{23}=L_{15}-\frac{\Delta}{2a_1}(L_{13}+L_{01})$$

$$J_{31}=L_{25}-\frac{\Delta}{2a_1}(L_{23}+L_{02})$$

$$J_{01}=L_{25}-\frac{\Delta}{2a_1}(L_{23}+L_{02})+\frac{a_1}{\Delta}(L_{23}-L_{02})$$

$$J_{02} = -L_{15} + \frac{\Delta}{2a_1}(L_{13} + L_{01}) - \frac{a_1}{\Delta}(L_{13} - L_{01})$$

$$J_{03} = -L_{03} + \frac{a_1}{\Delta}(L_{35} + L_{05}) \tag{2.1.4}$$

These satisfy the Lie Algebra of the group SO(3,1). So the group SO(3,1) is the group of invariance of equation (2.1.1.) for $\mu = 0$. We arrive at the same result with the rotation

$$\eta_0'' = \eta_5 - \frac{\Delta}{2a_1}\left(1 + \frac{2a_1^2}{\Delta^2}\right)\eta_0 - \frac{\Delta}{2a_1}\left(1 - \frac{2a_1^2}{\Delta^2}\right)\eta_3$$

$$\eta_3'' = \eta_5 - \frac{\Delta}{2a_1}(\eta_0 + \eta_3)$$

$$\eta_5'' = \eta_5 - \frac{a_1}{\Delta}(\eta_0 - \eta_3)$$

$$\eta_1'' = \eta_1, \qquad \eta_2'' = \eta_2 \tag{2.1.5}$$

For $p_\mu = (\omega, 0, 0, \omega)$ equation (2.13) becomes

$$\left\{\left(1 - \frac{\Delta^2}{z^2}\left[\frac{a_1}{\Delta}(iz_4 + z_3) + z_5\right]^2\right) \times \left[z^2\frac{\partial^2}{\partial z_i^2} - \left(z_i\frac{\partial}{\partial z_i}\right)^2 - 3z_i\frac{\partial}{\partial z_i} - 2\right] - \lambda\right\} Y(z_i) = 0 \tag{2.1.6}$$

The above equation, in the variables z_i'' defined by equations (2.11) and (2.1.5), becomes

$$\left[z''^2\frac{\partial^2}{\partial z_i''^2} - \left(z_i''\frac{\partial}{\partial z_i''}\right)^2 - 3z_i''\frac{\partial}{\partial z_i''} - 2 - \frac{\lambda}{1 - \Delta^2(z_5''^2/z''^2)}\right] Y(z_i'') = 0 \tag{2.1.7}$$

We easily see that equation (2.1.7) is invariant with respect to SO(4) transformations.†

Introducing polar coordinates

$$\begin{aligned} z_5 &= z\cos\vartheta_1 \\ z_4 &= z\sin\vartheta_1\cos\vartheta_2 \\ z_3 &= z\sin\vartheta_1\sin\vartheta_2\cos\vartheta_3 \\ z_2 &= z\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3\cos\phi \\ z_1 &= z\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3\sin\phi \end{aligned} \tag{2.1.8}$$

the partial differential operator of equation (2.1.7) takes the form (for simplicity we have dropped the primes)

$$z^2\frac{\partial^2}{\partial z_i^2} - \left(z_i\frac{\partial}{\partial z_i}\right)^2 - 3z_i\frac{\partial}{\partial z_i} = (1 - x^2)\frac{\partial^2}{\partial x^2} - 4x\frac{\partial}{\partial x} - \frac{\mathbf{L}^2 - \mathbf{K}^2}{1 - x^2}$$

$$x = \cos\theta_1 \tag{2.1.9}$$

† After this work had been completed we were informed that the SO(4) symmetry of the Wick rotated BS equation, with p_μ light-like has also been established by Seto (1968).

where

$$\mathbf{L}^2 - \mathbf{K}^2 = \frac{\partial^2}{\partial \vartheta_2{}^2} + 2 \cot \vartheta_2 \frac{\partial}{\partial \vartheta_2} + \frac{1}{\sin^2 \vartheta_2} \left[\frac{1}{\sin \vartheta_3} \frac{\partial}{\partial \vartheta_3} \left(\sin \vartheta_3 \frac{\partial}{\partial \vartheta_3} \right) + \frac{1}{\sin^2 \vartheta_3} \frac{\partial^2}{\partial \phi^2} \right] \tag{2.1.10}$$

is the Casimir operator of our symmetry group.

Separating variables

$$Y(z_i) = F(z)\, Y_{nlm}(\vartheta_2, \vartheta_3, \phi)\, G_n(\vartheta_1) \tag{2.1.11}$$

where Y_{nlm} is a 4-dimensional spherical harmonic satisfying

$$(\mathbf{L}^2 - \mathbf{K}^2)\, Y_{nlm} = (n^2 - 1)\, Y_{nlm} \tag{2.1.12}$$

equation (2.1.7) is reduced to the ordinary differential equation

$$\left[(1 - x^2) \frac{\partial^2}{\partial x^2} - 4x \frac{\partial}{\partial x} - \frac{n^2 - 1}{1 - x^2} - \frac{\lambda}{1 - \Delta^2 x^2} - 2 \right] G_n(x) = 0 \tag{2.1.13}$$

The boundary conditions are†

$$(1 - x^2)^{\frac{n+1}{2}} \left[1 - n - (1 - x^2) \frac{\partial}{\partial x} \right] G_n(x) \bigg|_{x=1} = 0$$

$$(1 - x^2)^{\frac{n+1}{2}} \left[1 - n + (1 - x^2) \frac{\partial}{\partial x} \right] G_n(x) \bigg|_{x=-1} = 0 \tag{2.1.14}$$

The function $F(z)$ is not determined from equation (2.1.7). Since $z = q_5 =$ const, it will be determined from the normalization of the BS wave function.

In the BS literature one usually finds the function $g_n(x)$

$$g_n(x) = (1 - x^2)^{\frac{n+1}{2}}\, G_n(x) \tag{2.1.15}$$

In the $g_n(x)$ variable, equation (2.1.13) becomes

$$\left[(1 - x^2) \frac{\partial^2}{\partial x^2} + 2(n - 1)\, x \frac{\partial}{\partial x} - \frac{\lambda}{1 - \Delta^2 x^2} - n(n - 1) \right] g_n(x) = 0 \tag{2.1.16}$$

with the boundary conditions

$$\left[n + (1 - x) \frac{\partial}{\partial x} \right] g_n(x) \bigg|_{x=1} = 0$$

$$\left[n - (1 - x) \frac{\partial}{\partial x} \right] g_n(x) \bigg|_{x=-1} = 0 \tag{2.1.17}$$

With the change of variable

$$y = 1 - x^2 \tag{2.1.18}$$

† Equations (2.1.14) have been obtained in a way similar to that followed by Kummer (1964).

equation (2.1.16) becomes

$$\left\{\frac{d^2}{dy^2}+\left(\frac{\frac{1}{2}}{y-1}+\frac{1-n}{y}\right)\frac{d}{dy} + \frac{[n(n-1)/4]y-[\lambda+(\Delta^2-1)n(n-1)]/4\Delta^2}{y(y-1)[y-(\Delta^2-1/\Delta^2)]}\right\}g_n(y)=0 \qquad (2.1.19)$$

known in the mathematical literature as Heun's equation (Heun, 1889). Its solutions, satisfying the boundary condition (2.1.17) are called Heun's functions. Unfortunately little is known about these functions and in particular it is not possible in general to find an analytic expression for the eigenvalues λ. For this reason in the next section we shall approach the eigenvalue problem by a new method starting directly from equation (2.1.13).

2.2. *Time-Like Case*

For p_μ time-like equation (2.13), in the rest frame $p_\mu=(\sqrt{(s)},0)$, becomes

$$\left\{\left[1-\frac{a_t^2-\Delta^2}{z^2}\left(i\frac{a_t}{\sqrt{(a_t^2-\Delta^2)}}z_4+\frac{1}{\sqrt{(a_t^2-\Delta^2)}}z_5\right)^2\right] \times\left[z^2\frac{\partial^2}{\partial z_i^2}-\left(z_i\frac{\partial}{\partial z_i}\right)^2-3z_i\frac{\partial}{\partial z_i}-2\right]-\frac{\lambda}{1-(s/4)}\right\}Y(z_i)=0$$

$$a_t=\sqrt{\left(\frac{s(1-\Delta^2)}{4-s}\right)} \qquad (2.2.1)$$

This, with the orthogonal transformation

$$iz_4''=i\frac{a_t}{\sqrt{(a_t^2-\Delta^2)}}z_4+\frac{\Delta}{\sqrt{(a_t^2-\Delta^2)}}z_5$$

$$z_5''=-i\frac{\Delta}{\sqrt{(a_t^2-\Delta^2)}}z_4-\frac{a_t}{\sqrt{(a_t^2-\Delta^2)}}z_5$$

$$z_1''=z_1,\qquad z_2''=z_2,\qquad z_3''=z_3 \qquad (2.2.2)$$

gives

$$\left[z''^2\frac{\partial^2}{\partial z_i''^2}-\left(z_i''\frac{\partial}{\partial z_i''}\right)^2-3z_i''\frac{\partial}{\partial z_i''}-2 -\frac{\lambda}{1-(s/4)+[(s/4)-\Delta^2](z_4''^2/z''^2)}\right]Y(z_i'')=0 \qquad (2.2.3)$$

It is clear that for the time-like case we have SO(4) symmetry both, in the original Minkowski space and after the Wick rotation in Euclidean space.

Following similar steps as in the light-like case we get

$$\left[(1-x^2)\frac{\partial^2}{\partial x^2}-4x\frac{\partial}{\partial x}-\frac{n^2-1}{1-x^2} - \frac{\lambda}{1-(s/4)+[(s/4)-\Delta^2]x^2}\right]G_n(x)=0 \qquad (2.2.4)$$

with $x \equiv (z_4/z)$ and the same boundary conditions.†

3. *Solutions of the Eigenvalue Equations*

As mentioned in the previous section we shall solve the eigenvalue problem with a new approach.

3.1. *Light-Like Region*

We start from

$$HG_\nu(x) = \nu^2 G_\nu(x) \qquad (3.1.1)$$

where H is the self-adjoint differential operator

$$H=(1-x^2)^2\frac{d^2}{dx^2}-4x(1-x^2)\frac{d}{dx}-2(\tfrac{1}{2}-x^2)+\frac{\lambda(1-x^2)}{1-\Delta^2 x^2} \qquad (3.1.2)$$

This can be considered as an eigenvalue equation for ν^2, which generalizes the spectrum of the Casimir operator $\mathbf{L}^2-\mathbf{K}^2+1$ of the symmetry group. For $\nu = n$ an integer number, equation (3.1.2) reduces to equation (2.6).

With the change of variables

$$\chi = \tfrac{1}{2}\ln\frac{1+x}{1-x},$$

$$\omega_\nu(\chi) = \sqrt{(1-x^2)}\, G_\nu(x)$$

equation (3.1.1) is put in the form

$$\left[-\frac{d^2}{d\chi^2}+V(\chi)\right]\omega(\chi) = -\nu^2\omega(\chi), \qquad (3.1.3)$$

where

$$V(\chi) = -\frac{\lambda}{1+(1-\Delta^2)\,sh^2\chi} \qquad (3.1.4)$$

This is a one-dimensional 'Schrödinger' equation for a particle of mass $\frac{1}{2}$ moving in the 'potential' $V(\chi)$. In Fig. 1 we have plotted $V(\chi)$ for λ positive and various values of Δ. The 'energy levels' correspond to the spectrum $-\nu^2$ which defines the Regge poles. It is clear that for λ sufficiently large, i.e. for strong attractive potential, there will always exist discrete levels. When ν becomes an integer n, which can happen only for special values of λ, the Regge poles materialize as physical bound states.

† For p_μ space-like we are led again to equation (2.2.4).

We now proceed to solve the 'Schrödinger' equation. First we approximate $V(\chi)$ by

$$V_{\rm app}(\chi) = -\frac{\lambda}{1 + sh^2\sqrt{(1-\Delta^2)}\,\chi} \tag{3.1.5}$$

For $\Delta = 0$, and $\Delta = 1$ this potential coincides with the exact one. As seen from Fig. 1 the approximation is very good for $0 \leqslant \Delta \lesssim 0{\cdot}75$ and becomes better and better as $\Delta \to 0$. Equation (3.1.3), with $V(\chi)$ replaced by $V_{\rm app}(\chi)$ can be solved exactly. We find†

$$\omega_{\rm app}(\chi) = \frac{U_{\rm app}[-sh^2\sqrt{(1-\Delta^2)}\,\chi]}{ch^{\sigma}\sqrt{(1-\Delta^2)}\,\chi}, \quad \text{where} \quad \sigma = \tfrac{1}{2}\left[\sqrt{\left(\frac{4\lambda}{1-\Delta^2}+1\right)}-1\right]$$

$$U_{\rm app}(\chi) = \begin{cases} F\left[-\frac{\kappa}{2}, \frac{-\nu}{\sqrt{(1-\Delta^2)}} - \frac{\kappa}{2}, \frac{1}{2}; -sh^2\sqrt{(1-\Delta^2)}\chi\right] & \kappa = 0, 2, 4, \ldots \\ sh\sqrt{(1-\Delta^2)}\,\chi\, F\left[-\frac{\kappa-1}{2}, \frac{-\nu}{\sqrt{(1-\Delta^2)}} - \frac{\kappa-1}{2}, \frac{3}{2}; -sh^2\sqrt{(1-\Delta^2)}\,\chi\right], & \kappa = 1, 3, 5, \ldots \end{cases} \tag{3.1.6}$$

and

$$\nu = \sqrt{(1-\Delta^2)}\,(\sigma - \kappa). \tag{3.1.7}$$

κ is the number of nodes of the wave functions.

For physical bound states ($\nu = n =$ integer) equation (3.1.7) gives the eigenvalue spectrum

$$\lambda_{n,\kappa} = [n + \kappa\sqrt{(1-\Delta^2)}]\,[n + (\kappa+1)\sqrt{(1-\Delta^2)}] \tag{3.1.8}$$

In the limit $\Delta^2 \to 1$, $V(\chi)$ of equation (3.1.4) is transformed into a square well potential of depth $-\lambda$ and range tending to infinity with $1/(1-\Delta^2)$. Alternatively, with a change in the scale, the range can be kept at a fixed value. For example, with a new variable $\chi' = (\pi/2)(\chi/\epsilon)$ and an equivalent scaling $4\epsilon^2/\pi^2$ in the mass, the range of the potential is kept at $\pi/2$. When ϵ, defined by

$$\epsilon = \ln\frac{4}{\sqrt{(1-\Delta^2)}}$$

is large the approximation becomes very good. The spectrum in this region is given as a solution of the equations

$$\sqrt{(\lambda-\nu^2)}\,\mathrm{tg}\,\epsilon\sqrt{(\lambda-\nu^2)} = \nu \tag{3.1.9a}$$

for even number of nodes, and

$$\sqrt{(\lambda-\nu^2)}\,\mathrm{cotg}\,\epsilon\sqrt{(\lambda-\nu^2)} = -\nu \tag{3.1.9b}$$

for odd numbers of nodes.

† See, for instance, Landau & Lifshitz (1958).

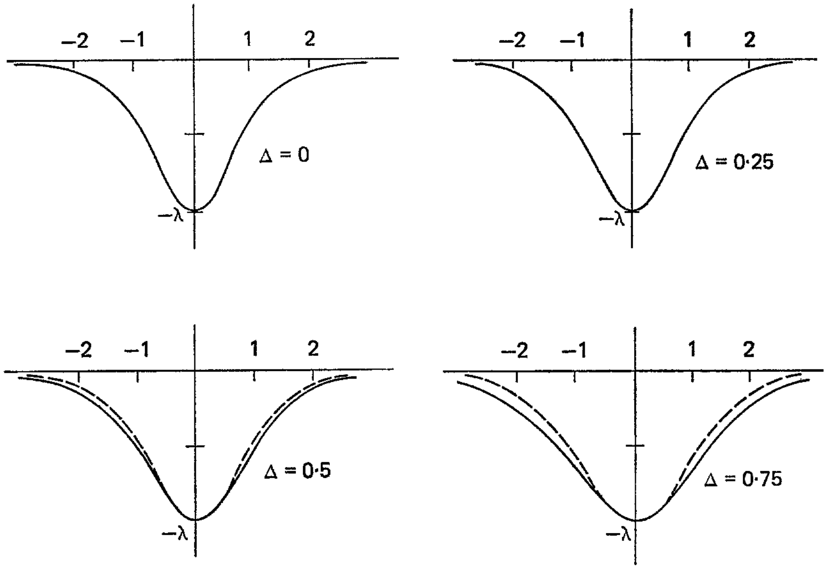

Figure 1.—The 'potential' for typical values of Δ. The solid lines refer to the exact potential

$$V(\chi) = -\frac{\lambda}{1 + (1 - \Delta^2) sh^2 \chi}$$

and the dashed lines to

$$V_{\mathrm{app}}(\chi) = -\frac{\lambda}{1 + sh^2 \sqrt{(1 - \Delta^2)}\, \chi}$$

For $\Delta = 0$ we have $V(\chi) = V_{\mathrm{app}}(\chi)$ and for $\Delta = 0{\cdot}25$ the difference $V(\chi) - V_{\mathrm{app}}(\chi)$ is not visible.

For physical bound states ($\nu = n =$ integer) equation (3.1.9), as $\Delta \to 1$, gives the eigenvalue spectrum

$$\lambda_n = n^2 + \frac{\frac{\pi^2}{4}(1 + \kappa)^2}{\left\{\ln \frac{4}{\sqrt{1 - \Delta^2}}\right\}^2}, \tag{3.1.10}$$

where $\kappa = 0, 1, 2, \ldots$.

3.2. *Time-Like Bound State Region*

Consider the time-like bound states for which $0 < s < 4m^2, s = (2m - E_b)^2$, where E_b is the binding energy. The time-like and the light-like eigenvalue problems are mathematically equivalent. Indeed the former can be reduced to the latter, i.e. to equation (3.1.1) of the light-like case by the simple substitution

$$\lambda \to \lambda_{\mathrm{eq}} = \frac{\lambda}{1 - (s/4)}, \qquad \Delta^2 \to \Delta^2_{\mathrm{eq}} = \frac{(s/4) - \Delta^2}{(s/4) - 1} \tag{3.2.1}$$

We consider two regions.

(i) $4\Delta^2 \leqslant s \leqslant 4$

In this region

$$0 \geqslant \Delta_{eq} \geqslant -\infty \qquad \text{and} \qquad \frac{\lambda}{1-\Delta^2} \leqslant \lambda_{eq} \leqslant \infty$$

The solutions and the spectrum of the eigenvalue equation are obtained by substituting (3.2.1) into (3.1.6) and (3.1.7). We find

$$\lambda^{app}_{n,\kappa} = \left(1-\frac{s}{4}\right)\left[n+\kappa\sqrt{\left(\frac{1-\Delta^2}{1-(s/4)}\right)}\right]\left[n+(\kappa+1)\sqrt{\left(\frac{1-\Delta^2}{1-(s/4)}\right)}\right]$$
$$\kappa = 0, 1, 2, \ldots \qquad (3.2.2)$$

As we have discussed previously, equation (3.2.2) is a very good approximation for $0 \leqslant |\Delta_{eq}| \lesssim 3$, which means $4\Delta^2 \leqslant s \lesssim 3+\Delta^2$. For $s = 4\Delta^2$, equation (3.2.2) gives us the exact spectrum.

In the nonrelativistic limit $(E_b/2) \to 0$, the 'potential' takes the form

$$V(\chi) \sim -c\,\delta(\chi),$$

$$c = \frac{2\lambda}{\sqrt{[(1-\Delta^2)E_b]}}\operatorname{arctg}\sqrt{\left(\frac{1-\Delta^2}{E_b}\right)} \approx \frac{\pi\lambda}{\sqrt{[(1-\Delta^2)E_b]}}$$

and we have

$$\omega_\nu(\chi) = \exp(-\nu|\chi|) \qquad (3.2.3)$$

$$\nu = \frac{\pi}{2}\frac{\lambda}{\sqrt{[(1-\Delta^2)E_b]}}$$

This corresponds to the nonrelativistic limit of the normal solutions ($\kappa = 0$). For the abnormal ones ($\kappa \neq 0$), we have $\lambda \to \frac{1}{4}$ (Wick, 1954; Cutkosky, 1954).

(ii) $0 \leqslant s \leqslant 4\Delta^2$

In this region we have

$$\lambda \leqslant \lambda_{eq} \leqslant \frac{\lambda}{1-\Delta^2}, \qquad \text{and} \qquad 1 > \Delta^2 \geqslant \Delta^2_{eq} \geqslant 0$$

For $s = 4\Delta^2$ we have the exact solution, as we mention previously. In general our approximate solutions, very similar to the light-like ones, are extremely good when $s \approx 4\Delta^2$, and as $s \to 0$ they tend smoothly to the solution of the light-like case.

3.3 *The Space-Like Region*

For $-\infty < s < 0$ we have $0 < \lambda_{eq} < \lambda$, and $1 > \Delta^2_{eq} > \Delta^2$. We obtain

$$\nu_{app} = \frac{1}{2}\frac{1}{\sqrt{[1-(s/4)]}}[\sqrt{(4\lambda+1-\Delta^2)} - (2\kappa+1)\sqrt{(1-\Delta^2)}] \qquad (3.3.1)$$

a good approximation when $0 \leqslant \Delta \lesssim 0{\cdot}75$ and $[\Delta^2-(s/4)]/[1-(s/4)] \leqslant \frac{1}{2}$.

As $\Delta \to 1$ we have the asymptotic form

$$\nu = \left\{\frac{\lambda}{1-(s/4)} - \frac{\pi^2}{8}(1+\kappa)^2\left[\ln\frac{16[1-(s/4)]}{1-\Delta^2}\right]^{-2}\right\}^{1/2} \tag{3.3.2}$$

We remark finally that as $s \to -\infty$, equation (3.3.2) gives the asymptotic form of ν for all Δ.

Note Added in Proof

After completing this paper related work done by Seto (1969) was pointed to us. In reference to this we add the following remarks:

(1) We treat the problem of symmetry in a different way.

(2) In our approach the radial equation is attached directly as a Regge trajectory eigenvalue problem. (Usually the eigenvalue is the coupling constant.) In this way we are led to a perturbation series in $\Delta_{eq}^2 = (s/4-\Delta^2)/(s/4-1)$ instead of $(s/4-\Delta^2)/(\Delta^2-1)$ in Nakanishi (1965) and Seto (1969). For example, for the Regge trajectories we have $\nu = \sum_{l=0}^{\infty} \alpha_l \Delta_{eq}^{2l}$. Our expansion parameter Δ_{eq}^2 is smaller than one in the whole half of the complex s-plane, $\text{Re}\, s < 2(1+\Delta^2)$ while in the Nakanishi–Seto case, the expansion parameter is smaller than one only inside the circle $|s-4\Delta^2| < 4(1-\Delta^2)$.

The constant terms, which give the exact position of the trajectories at $s = 4\Delta^2$, are the same in both expansions of course. Also the coefficients of the first-order terms, which give the slope of the trajectories at $s = 4\Delta^2$, are the same

$$\nu^2 = \nu_0{}^2 + \lambda_{eq}\Delta_{eq}^2 \frac{2\nu_0(2\kappa^2+4\kappa\nu_0+2\kappa+2\nu_0-\lambda)}{(2\kappa+2\nu_0+3)(2\kappa+2\nu_0+1)(2\kappa+2\nu_0-1)} + 0(\Delta_{eq}^4)$$
$$\nu_0 = [\sqrt{(\lambda+1/4)} - \kappa - 1/2]$$

(3) We give also good approximative expressions in closed form (non-perturbative), for both the eigenvalues and the eigenfunctions.

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